

One-Loop Radiative Corrections to the QED Casimir Energy

Amirhosein Mojavezi and Reza Moazzemi*



Department of Physics, University of Qom, Ghadir Blvd., Qom 371614-611, Iran

In this paper, we investigate one-loop radiative corrections to the Casimir energy in the presence of two perfectly conducting parallel plates for QED theory within the renormalized perturbation theory. In fact, there are three contributions for radiative correction to the Casimir energy, up to order α . Only the main term, which is two-loop, has been computed by Bordag et. al (1985). Here, we consider corrections due to two remaining terms, i.e., photonic and fermionic one-loop correction resulting from renormalized QED Lagrangian. Our results show that only the fermionic loop has a minor correction.

I. INTRODUCTION

The Casimir effect is the physical manifestation of the change in the zero point energy of a quantum field for different configurations. The zero point configuration refers to one in which there does not exist any on-shell physical excitation of the field.

In 1948 Casimir predicted the existence of this effect as an attractive force between two infinite parallel uncharged perfectly conducting plates in vacuum [1]. This effect was subsequently observed experimentally by Sparnaay in 1958 [2] (for a general review on the Casimir effect, see Refs.[3, 4]). Similar measurements have been done for other geometries, and their precisions have been greatly improved [5–10]. The manifestations of the Casimir effect have been studied in many different areas of physics. For example, the magnitude of the cosmological constant has been estimated using the Casimir effect [11–13]. This effect has been also studied within the context of string theory [14]. It has been investigated in connection with the properties of the space-time with extra dimensions [15–17]. The majority of the investigations related to the Casimir effect concerns with the calculation of this energy or the consequence forces for different fields in different geometries, such as parallel plates [1, 18], cubes [19–27], cylinders [26, 28–30], and spherical geometries [26, 31–33].

Although the Casimir effect has been known for nearly 60 years, the question of what are the leading radiative corrections to this effect is still surprisingly a subject of discussion. The first endeavors to compute the radiative corrections to the Casimir energy were reported in a paper by Bordag, Robaschik, and Wieczorek (BRW) [34]. There exist many works on the radiative corrections to the Casimir energy for various cases (see for example [34–44]). In the case of a real massive scalar field, Next to Leading Order (NLO) correction to the energy has been computed in [3, 27, 40–49]. Moreover, the two-loop radiative corrections for some effective field theories have been investigated in [36–38]. Bordag and his collaborators have calculated radiative correction to the Casimir energy due to one of the three related terms of order of α , , in the presence of two perfectly conducting parallel plates for QED theory. In this view point, the photon propagator satisfies boundary conditions on the plates, while the plates are not transparent to the electrons. They found the correction $E_0^{(1)} = \frac{\pi^2 \alpha}{2560 m a^4}$ to the popular leading term of Casimir energy $E_0^{(0)} = -\frac{\pi^2}{720 a^3}$, where a is the distance between plates and m is the electron mass. In 1998 this result with another approach has been reported [35]. In the framework of renormalized perturbation theory for QED there are three vacuum bubbles of order of α . Up to now, all the papers on the Casimir effect that we are aware of, have not been calculated two of these diagrams to the radiative corrections, namely: photonic loop  resulting from

Electromagnetic field and fermionic loop  from spinor field.

The primary purpose of this paper is to directly calculate radiative correction to the Casimir energy resulting from one-loop corrections namely: one-loop photon and one-loop fermion, in the framework of the renormalized perturbation theory for QED theory. These corrections are of order α . In order to do this, we use Green's functions in the presence of plates for Electromagnetic field with Dirichlet boundary condition and for spinor field with MIT bag boundary condition as propagators. Our main regularization is dimensional regularization. However, during the calculations we need cut-off regularization to eliminate infinities.

*Electronic address: r.moazzemi@qom.ac.ir

Our approach in the calculation of radiative corrections to Casimir energy is the most direct one. In this way we subtract two infinite energies: one relate to presence and the other without presence of two plates. We justify both of them to such a way our physical result is obtained.

We organized the paper as follows: In Section II we review briefly renormalization of Quantum Electrodynamics. In Section III using analogies between an Electromagnetic field and a massless scalar field, photonic loop correction is considered. We use the Dirichlet boundary condition on the two plates. In Section IV we directly calculate radiative correction to the Casimir energy resulting from fermionic loop where MIT bag boundary condition as constraints on both of the plates is considered. In Section 5 we summarize our results and state our conclusions.

II. RENORMALIZATION OF QUANTUM ELECTRODYNAMICS

In this section we briefly review systematics of renormalization for QED theory (see for complete details [50]). The original QED Lagrangian is

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\rlap{\not{\partial}} - m_0)\psi - e\bar{\psi}\gamma_\mu\psi A^\mu. \quad (1)$$

By replacing $\psi = z_2^{\frac{1}{2}}\psi_r$ and $A^\mu = z_3^{\frac{1}{2}}A_r^\mu$, it becomes

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}z_3(F_{\mu\nu}^r)^2 + z_2\bar{\psi}_r(i\rlap{\not{\partial}} - m_0)\psi_r - e_0z_2z_3^{\frac{1}{2}}\bar{\psi}_r\gamma_\mu\psi_r A_r^\mu, \quad (2)$$

where e is the physical electric charge and z_2 and z_3 are the field-strength renormalizations for ψ and A^μ respectively. We define a scaling factor z_1 as follows:

$$ez_1 = e_0z_2z_3^{\frac{1}{2}}. \quad (3)$$

We can split each term of the Lagrangian into two pieces as follows:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}(F_{\mu\nu}^r)^2 + \bar{\psi}_r(i\rlap{\not{\partial}} - m)\psi_r - e\bar{\psi}_r\gamma_\mu\psi_r A_r^\mu - \frac{1}{4}\delta_3(F_{\mu\nu}^r)^2 + \bar{\psi}_r(i\delta_2\rlap{\not{\partial}} - \delta_m)\psi_r - e\delta_1\bar{\psi}_r\gamma_\mu\psi_r A_r^\mu, \quad (4)$$

where $\delta_3 = z_3 - 1$, $\delta_2 = z_2 - 1$, $\delta_m = z_2m_0 - m$ and $\delta_1 = z_1 - 1 = (\frac{e_0}{e})z_2z_3^{\frac{1}{2}} - 1$ are counterterms and m is the physical

mass of the electron. Now, the Feynman rules for this Lagrangian are

$$\begin{array}{c} \mu \\ \text{---} \bullet \text{---} \\ \text{---} \end{array} = -ie\gamma^\mu \quad (5)$$

$$\begin{array}{c} \mu \\ \text{---} \otimes \text{---} \\ \text{---} \end{array} = -ie\delta_1\gamma^\mu \quad (6)$$

$$\mu \sim \text{---} \xrightarrow{k} \nu = \frac{-i}{k^2 + i\epsilon} \left(g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right) \quad (7)$$

$$\mu \sim \text{---} \otimes \text{---} \nu = -i(g^{\mu\nu}k^2 - k^\mu k^\nu)\delta_3 \quad (8)$$

$$\text{---} \xrightarrow{p} = \frac{i}{p - m + i\epsilon} \quad (9)$$

$$\text{---} \otimes \text{---} = i(p\delta_2 - \delta_m). \quad (10)$$

Each of the four counterterms must be fixed by renormalization conditions. For QED theory these conditions are (see for instance [50])

$$\mu \sim \text{---} \textcircled{1\text{PI}} \text{---} \nu = i\Pi^{\mu\nu}(q) = i(g^{\mu\nu}k^2 - k^\mu k^\nu)\Pi(k^2) \quad (11)$$

$$\text{---} \textcircled{1\text{PI}} \text{---} = -i\Sigma(p) \quad (12)$$

$$\left(\begin{array}{c} \mu \\ \text{---} \bullet \text{---} \\ \text{---} \end{array} \right)_{\text{amputated}} = -ie\Gamma^\mu(p', p). \quad (13)$$

In the above equations $i\Pi(k^2)$ defines the sum of all 1PI (1-particle-irreducible) insertions into the photon propagator. Up to leading order in α it becomes

$$\Pi(k^2) = \frac{-e^2}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{\left(m^2 - x(1-x)k^2\right)^{2 - \frac{d}{2}}} 8x(1-x). \quad (14)$$

Moreover, $-i\Sigma(p)$ denotes the sum of all 1PI diagrams with two external fermion lines. To leading order in α it is

$$-i\Sigma(p) = -i \frac{e^2 m}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{\left((1-x)^2 m^2 + x\mu^2 - x(1-x)p^2\right)^{2-\frac{d}{2}}} [(4-\epsilon)m - (2-\epsilon)xp], \quad (15)$$

with $\epsilon = 4 - d$. Also, $\Gamma^\mu(p', p)$ denotes the sum of vertex diagrams, more accurately

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(k^2) + \frac{i\sigma^{\mu\nu} k_\nu}{2m} F_2(k^2), \quad (16)$$

with F_1 and F_2 are unknown functions of k^2 called form factors and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. To lowest order, $F_1 = 1$ and $F_2 = 0$, we have $\Gamma^\mu = \gamma^\mu$. By using Eqs. (14), (15) and (16), up to leading order in α , the counterterms are derived as follows:

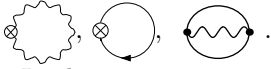
$$\delta_3 = \frac{-e^2}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{(m^2)^{2-\frac{d}{2}}} 8x(1-x), \quad (17)$$

$$\delta_m = \frac{-e^2 m}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{\left((1-x)^2 m^2 + x\mu^2\right)^{2-\frac{d}{2}}} (4 - 2x - \epsilon(1-x)), \quad (18)$$

$$\delta_2 = \frac{-e^2}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \frac{\Gamma(2 - \frac{d}{2})}{\left((1-x)^2 m^2 + x\mu^2\right)^{2-\frac{d}{2}}} \left((2-\epsilon)x - \frac{\epsilon}{2} \frac{2x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2} (4 - 2x - \epsilon(1-x)) \right), \quad (19)$$

$$\delta_1 = \frac{-e^2}{(4\pi)^{\frac{d}{2}}} \int_0^1 dz (1-z) \left(\frac{\Gamma(2 - \frac{d}{2})}{((1-z)^2 m^2 + z\mu^2)^{2-\frac{d}{2}}} \frac{(2-\epsilon)^2}{2} \right. \quad (20)$$

$$\left. + \frac{\Gamma(3 - \frac{d}{2})}{((1-z)^2 m^2 + z\mu^2)^{3-\frac{d}{2}}} [2(1-4z+z^2) - \epsilon(1-z)^2] m^2 \right), \quad (21)$$

where μ is a photon mass to control infrared divergences which will finally go to the zero. According to the above discussion three vacuum bubbles contribute to the Casimir energy: . Two first diagrams arise from Eqs. (8) and (10). Bordag et. al have computed the last one. In the next sections we will consider the other two.

III. PHOTONIC LOOP

In this section, we calculate NLO radiative correction to the Casimir energy due to the photonic loop. We use Dirichlet boundary condition on the two parallel perfectly conducting plates in (3+1) dimensions. Obviously in the presence of the two plates, propagators automatically incorporate the boundary conditions and are position dependent. The contribution of one-loop photon to the vacuum energy in the interval $[-\frac{a}{2}, \frac{a}{2}]$ is

$$\Delta E_{\text{Ph}} = \int_{-a/2}^{a/2} d^3x \langle \Omega | \mathcal{H}_I | \Omega \rangle = i \int_{-a/2}^{a/2} \text{diagram} d^3x + O(\alpha^2), \quad (22)$$

using Eq.(8) it becomes

$$\Delta E_{\text{Ph}}^{(1)} = i \int_{-a/2}^{a/2} d^3x G_B(x, x) [-i(g^{\mu\nu} k^2 - k^\mu k^\nu) \delta_3], \quad (23)$$

where $G_B(x, x')$ is the propagator of Electromagnetic field in the bounded space. For overall consistency, we use dimensional regularization to control ultraviolet divergences, and a photon mass μ to control infrared divergences.

Using analogies between an Electromagnetic field and a massless scalar field, photon propagator is considered as

$$G_B(x, x') = \frac{-2ig_{\mu\nu}}{a} \int \frac{dw}{2\pi} \int \frac{d^{d-2}k_{\perp}}{(2\pi)^{d-2}} \sum_n \frac{e^{-iw(t-t')} e^{-ik_{\perp} \cdot (x^{\perp} - x'^{\perp})} \sin(k_n(z + \frac{a}{2})) \sin(k_n(z' + \frac{a}{2}))}{w^2 - k_{\perp}^2 - k_n^2 + \mu^2}. \quad (24)$$

Here k_{\perp} and k_n denote the parallel and the perpendicular momenta to plates (in z -direction), respectively. Note that, both contributions related to TE mode and TM mode are considered to be the same, hence the final energy should be became twice. After the usual Wick rotation and using Eqs.(17) and (24) with $x = x'$ and carrying out the integration over the space then over solid angle in the d -dimensional Euclidean space we have

$$\Delta E_{\text{Ph}}^{(1)} = \frac{12L^{d-2}\delta_3\pi^{\frac{d-1}{2}}}{(2\pi)^{d-1}\Gamma(\frac{d-1}{2})} \int dk_E k_E^{d-2} \sum_n \frac{k_E^2 + k_n^2}{k_E^2 + k_n^2 + \mu^2}, \quad (25)$$

where $k_E^2 = w^2 + k_{\perp}^2$ and using the Dirichlet boundary condition on the walls

$$k_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \dots \quad (26)$$

The one-loop photon correction to the vacuum energy in free space is

$$\Delta E_{\text{Ph}}'^{(1)} = i \int d^3x G_F(x, x) [-i(g^{\mu\nu}k^2 - k^{\mu}k^{\nu})\delta_3], \quad (27)$$

where $G_F(x, x')$ is the propagator of Electromagnetic field in free space in Feynman gauge ($\xi = 1$). After carrying out the first integrations and multiplying by $\frac{a}{L}$ factor we have

$$\Delta E_{\text{Ph}}'^{(1)} = \frac{12aL^{d-2}\delta_3\pi^{\frac{d-1}{2}}}{(2\pi)^d\Gamma(\frac{d-1}{2})} \int dk_E k_E^{d-2} \int dk \frac{k_E^2 + k^2}{k_E^2 + k^2 + \mu^2}. \quad (28)$$

Changing of variable as $k' = \frac{ak}{2\pi}$ and integrating of k_E then expanding these terms about $d = 4$, all of the divergent terms should be eliminated by appropriately adjusting our regulators, for both free and bounded cases. Finally by integrating of x we obtain

$$\begin{aligned} \Delta E_{\text{Ph}}^{\text{Cas.}} = \Delta E_{\text{Ph}}^{(1)} - \Delta E_{\text{Ph}}'^{(1)} + O(\alpha^2) &= \frac{3e^2L^2\mu^2}{4\pi^2} \left[\sum_{n=1}^{\infty} \sqrt{n^2 + \frac{a^2\mu^2}{\pi^2}} \left(-2 + 2\gamma + \ln\left(\frac{n^2 + \frac{a^2\mu^2}{\pi^2}}{(\frac{2a}{Lm})^2}\right) \right) \right. \\ &\quad \left. - \int_0^{\infty} dk' \sqrt{k'^2 + \frac{a^2\mu^2}{\pi^2}} \left(-2 + 2\gamma + \ln\left(\frac{k'^2 + \frac{a^2\mu^2}{\pi^2}}{(\frac{2a}{Lm})^2}\right) \right) \right] + O(\alpha^2). \end{aligned} \quad (29)$$

Now, we can use the Abel-Plana Summation Formula (APSF)[51], which basically converts a summation into an integration,


$$\sum_{n=1}^{\infty} f(n) = -\frac{f(0)}{2} + \int_0^{\infty} dx f(x) + i \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} [f(it) - f(-it)]. \quad (30)$$

We apply this formula for Eq.(29) to get (see Appendix for details)

$$\begin{aligned} \Delta E_{\text{Ph}}^{\text{Cas.}} &= \frac{-3e^2L^2\mu^3}{4\pi^3a} \left\{ -2 + 2\gamma + \ln\left(\frac{L^2m^2\mu^2}{4\pi^2}\right) \right. \\ &\quad \left. + \frac{2}{\mu} \int_0^{\infty} dt \frac{\sqrt{\pi^2t^2 - a^2\mu^2}}{e^{2\pi t} - 1} \left[-2 + 2\gamma + \ln\left(\frac{L^2m^2}{4a^2\pi^2}(\pi^2t^2 - a^2\mu^2)\right) \right] \right\} + O(\alpha^2). \end{aligned} \quad (31)$$

It is obvious that as μ goes to zero, photonic loop does not have any contribution to $O(\alpha)$ radiative correction to the Casimir energy.

IV. FERMIONIC LOOP

In this section, we calculate radiative correction to the Casimir energy due to fermionic loop . We use the MIT bag boundary condition on the plates. According to MIT bag boundary condition there is no flux of fermions through the boundary, this means that

$$n_\mu j^\mu = 0, \quad (32)$$

where j^μ indicates the current of the Dirac field and n_μ is the normal unit vector to the boundary, or more strictly it implies to complete confinement of the spinor field. Then MIT bag boundary condition turns out to be [53–55]

$$(1 + i(\hat{n} \cdot \vec{\gamma}))\psi(\vec{x}) = 0, \quad (33)$$

which is satisfied on the boundary, more accurate on the two plates. Applying this condition to Dirac spinor field, one can derive

$$pa \cot(pa) = -ma, \quad (34)$$

which determines quantized modes. Two limits are interesting to calculate, small mass and large mass limits. For small mass limit the solutions of Eq.(34) are (see for more details [56])

$$p_n = (n + \frac{1}{2})\frac{\pi}{a} \quad \text{with } n = 0, 1, 2, \dots \quad (35)$$

where p_n denotes the parallel momenta to the plates (in z -direction). Now, for the bounded space we have

$$\Delta E_F = \int_{-a/2}^{a/2} d^3x \langle \Omega | \mathcal{H}_I | \Omega \rangle = i \int_{-a/2}^{a/2} d^3x \otimes \text{loop} \quad d^3x + O(\alpha^2) = i \int_{-a/2}^{a/2} d^3x S_B(x, x) [i(p\delta_2 - \delta_m)] + O(\alpha^2), \quad (36)$$

where $S_B(x, x')$ is the propagator of spinor field between plates [55, 56]

$$S_B(x, x') = \frac{i}{a} \int \frac{dw}{2\pi} \int \frac{d^{d-2}p_\perp}{(2\pi)^{d-2}} \sum_{n=0} \frac{p+m}{w^2 - p_\perp^2 - p_n^2 - m^2} e^{-iw(t-t')} e^{-ip_\perp(x^\perp - x'^\perp)} e^{-ip_n(z-z')}. \quad (37)$$

Here p_\perp and p_n indicate the parallel and the perpendicular momenta to the plates, respectively. After the usual Wick rotation and Carrying out the integration, one can obtain

$$\Delta E_F^{(1)} = 8L^{L-2} \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \frac{1}{(2\pi)^{d-1}} \int p_E^{d-2} \sum_{n=0} \left(\frac{(p_E^2 + p_n^2)\delta_2}{p_E^2 + p_n^2 + m^2} + \frac{m\delta_m}{p_E^2 + p_\perp^2 + p_n^2 + m^2} \right), \quad (38)$$

where $p_E^2 = w^2 + p_\perp^2$. Similarly, for the free space we have

$$\Delta E_F'^{(1)} = i \int d^3x S_B(x, x) [i(p\delta_2 - \delta_m)]. \quad (39)$$

Using Eq.(9) we obtain

$$\Delta E_F'^{(1)} = \frac{16aL^{d-2}}{(2\pi)^{d-1}} \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int dp_E p_E^{d-2} \int \frac{dp}{2\pi} \left(\frac{(p_E^2 + p^2)\delta_2}{p_E^2 + p^2 + m^2} + \frac{m\delta_m}{p_E^2 + p^2 + m^2} \right). \quad (40)$$

For the massless case the radiative correction to Casimir energy due to fermionic loop is

$$\begin{aligned}
\Delta E_F^{\text{Cas.}} &= \Delta E_F^{(1)} - \Delta E_F'^{(1)} + O(\alpha^2) \\
&= \frac{8L^2\pi^{\frac{d-1}{2}}}{(2\pi)^{d-1}\Gamma(\frac{d-1}{2})} \int dp_E p_E^{d-2} \left[\left(\sum_{n=0}^{\infty} \frac{((n+\frac{1}{2})^2 + \frac{a^2 p_E^2}{\pi^2})\delta_2}{(n+\frac{1}{2})^2 + \frac{a^2 p_E^2}{\pi^2} + \frac{a^2 m^2}{\pi^2}} - \int dp' \frac{(p'^2 + \frac{a^2 p_E^2}{\pi^2})\delta_2}{p'^2 + \frac{a^2 p_E^2}{\pi^2} + \frac{a^2 m^2}{\pi^2}} \right) \right. \\
&\quad \left. + \left(\frac{a^2 m \delta_m}{\pi^2} \right) \left(\sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2 + \frac{a^2 p_E^2}{\pi^2} + \frac{a^2 m^2}{\pi^2}} - \int dp' \frac{1}{p'^2 + \frac{a^2 p_E^2}{\pi^2} + \frac{a^2 m^2}{\pi^2}} \right) \right] + O(\alpha^2). \tag{41}
\end{aligned}$$

Changing of variables as $p' = \frac{a\pi}{2a}$ and integrating of p_E then expanding these terms about $d = 4$ and $\mu = 0$, as mentioned before, all of the divergent terms by appropriately adjusting our regulators, are eliminated. Finally, by integrating of x we obtain

$$\begin{aligned}
\Delta E_F^{\text{Cas.}} &= \frac{e^2 L^2 m^2}{16\pi^2 a} \left\{ \sum_{n=0}^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + (n+\frac{1}{2})^2} \left[-6 + 2\gamma + \ln \left(\frac{\frac{a^2 m^2}{\pi^2} + (n+\frac{1}{2})^2}{(\frac{2a}{m^3})^2} \right) \right] \right. \\
&\quad - \int_0^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + p'^2} \left[-6 + 2\gamma + \ln \left(\frac{\frac{a^2 m^2}{\pi^2} + p'^2}{(\frac{2a}{m^3})^2} \right) \right] \\
&\quad + \sum_{n=0}^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + (n+\frac{1}{2})^2} \left[10 - 6\gamma + 6 \ln 2 - 3 \ln \left(\frac{\frac{a^2 m^2}{\pi^2} + (n+\frac{1}{2})^2}{(\frac{a\pi}{m^3})^2} \right) \right] \\
&\quad \left. - \int_0^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + p'^2} \left[10 - 6\gamma + 6 \ln 2 - 3 \ln \left(\frac{\frac{a^2 m^2}{\pi^2} + p'^2}{(\frac{a\pi}{m^3})^2} \right) \right] \right\} + O(\alpha^2).
\end{aligned}$$

Here we need another type of APSF to convert the sum into integral,

$$\sum_{n=0}^{\infty} f(n+\frac{1}{2}) = \int_0^{\infty} dx f(x) - i \int_0^{\infty} \frac{dt}{e^{2\pi t} + 1} [f(it) - f(-it)]. \tag{42}$$

Applying this formula in Eq.(42) (see Appendix for more details) it becomes

$$\Delta E_F^{\text{Cas.}} = \int_{\frac{am}{\pi}}^{\infty} dt \frac{e^2 L^2 m^2 \sqrt{-a^2 m^2 + \pi^2 t^2}}{8a\pi^2 (e^{2\pi t} + 1)} \left(4 - 4\gamma - 2 \ln \left(\frac{-a^2 m^2 + \pi^2 t^2}{(2a\pi)^2} \right) \right) + O(\alpha^2). \tag{43}$$

To calculate this expression we use the following summation formula:

$$\sum_{n=1}^{\infty} (-1)^{n+1} e^{-2\pi n t} = \frac{1}{e^{2\pi t} + 1}. \tag{44}$$

The final result of the radiative correction to the Casimir energy due to fermionic loop, for the small mass case, is obtained as follows:

$$\Delta E_F^{\text{Cas.}} = \sum_{n=1}^{\infty} \frac{(-1)^n e^2 L^2 m^2}{16a n^2 \pi^2} \left(K_0(2nam) + 2nam K_1(2nam) \left(\gamma - 2 \ln \pi - \ln \left(\frac{16nam}{m^2} \right) \right) \right). \tag{45}$$

The other interesting limit is the large mass limit. In this case, Eq.(34) turns out to be

$$ma \tan(pa) = -pa. \tag{46}$$

Now, the solutions are

$$p_n = \frac{n\pi}{a} \text{ with } n = 1, 2, \dots \tag{47}$$

The radiative correction to Casimir energy in this case, using Eqs. (38) and (40), becomes

$$\begin{aligned}\Delta E_F^{\text{Cas.}} &= \Delta E_F^{(1)} - \Delta E_F'^{(1)} + O(\alpha^2) \\ &= \frac{8L^2\pi^{\frac{d-1}{2}}}{(2\pi)^{d-1}\Gamma(\frac{d-1}{2})} \int dp_E p_E^{d-2} \left[\left(\sum_{n=1}^{\infty} \frac{(n^2 + \frac{a^2 p_E^2}{\pi^2}) \delta_2}{n^2 + \frac{a^2 p_E^2}{\pi^2} + \frac{a^2 m^2}{\pi^2}} - \int dp' \frac{(p'^2 + \frac{a^2 p_E^2}{\pi^2}) \delta_2}{p'^2 + \frac{a^2 p_E^2}{\pi^2} + \frac{a^2 m^2}{\pi^2}} \right) \right. \\ &\quad \left. + \frac{a^2 m \delta_m}{\pi^2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2 + \frac{a^2 p_E^2}{\pi^2} + \frac{a^2 m^2}{\pi^2}} - \int dp' \frac{1}{p'^2 + \frac{a^2 p_E^2}{\pi^2} + \frac{a^2 m^2}{\pi^2}} \right) \right] + O(\alpha^2).\end{aligned}\quad (48)$$

In a similar way for obtaining Eq. (42), now we have

$$\begin{aligned}\Delta E_F^{\text{Cas.}} &= \frac{e^2 L^2 m^2}{16\pi^2 a} \left[\left(\sum_{n=1}^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + n^2} \left(-6 + 2\gamma + \ln\left(\frac{\frac{a^2 m^2}{\pi^2} + n^2}{(\frac{2a}{m^3})^2}\right) \right) \right. \right. \\ &\quad \left. \left. - \int_0^\infty \sqrt{\frac{a^2 m^2}{\pi^2} + p'^2} \left(-6 + 2\gamma + \ln\left(\frac{\frac{a^2 m^2}{\pi^2} + p'^2}{(\frac{2a}{m^3})^2}\right) \right) \right) \right. \\ &\quad \left. + \left(\sum_{n=1}^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + n^2} \left(10 - 6\gamma + 6 \ln 2 - 3 \ln\left(\frac{\frac{a^2 m^2}{\pi^2} + n^2}{(\frac{a\pi}{m^3})^2}\right) \right) \right. \right. \\ &\quad \left. \left. - \int_0^\infty \sqrt{\frac{a^2 m^2}{\pi^2} + p'^2} \left(10 - 6\gamma + 6 \ln 2 - 3 \ln\left(\frac{\frac{a^2 m^2}{\pi^2} + p'^2}{(\frac{a\pi}{m^3})^2}\right) \right) \right) \right] + O(\alpha^2).\end{aligned}\quad (49)$$

Here we can apply the APSF, Eq. (30), as we have used in photonic loop, for the above expression, to obtain

$$\Delta E_F^{\text{Cas.}} = -\frac{e^2 L^2 m^3}{32\pi^3} \left(4 - 4\gamma + 2 \ln\left(\frac{32\pi^5}{m^8}\right) \right) + \int_{\frac{am}{\pi}}^\infty dt \frac{e^2 L^2 m^2 \sqrt{-a^2 m^2 + \pi^2 t^2}}{8a\pi^2 (e^{2\pi t} - 1)} \left(4 - 4\gamma - 2 \ln\left(\frac{-a^2 m^2 + \pi^2 t^2}{(2a\pi)^2}\right) \right) + O(\alpha^2).\quad (50)$$

The first term, which appears due to the zero term in APSF, is independent of distance between plates a . Therefore this term has no impact on the physics of problem. Ignoring the first term and using the following equation:

$$\sum_{n=1}^{\infty} e^{-2\pi n t} = \frac{1}{e^{2\pi t} - 1},\quad (51)$$

we have

$$\Delta E_F^{\text{Cas.}} = -\sum_{n=1}^{\infty} \frac{(-1)^n e^2 L^2 m^2}{16a n^2 \pi^2} \left(K_0(2nam) + 2nam K_1(2nam) \left(\gamma - 2 \ln \pi - \ln\left(\frac{16nam}{m^2}\right) \right) \right).\quad (52)$$

Since the photonic correction vanishes up to order α , Eqs. (45) and (52) are, in fact, our corrections to the well-known BRW result for the small mass and large mass limits, respectively. We have compared our results with BRW one in Fig. 1. This figure shows that one loop correction due to photonic and fermionic loops is comparable with BRW result only when the distance between plates is about one to thirty times of Compton wavelength of fermion.

V. CONCLUSIONS

We have calculated one-loop radiative correction to the Casimir energy due to photonic and fermionic counterterms within the renormalized perturbation theory for QED theory. The topology considered here is two perfectly conducting parallel plates in (3+1) dimensions. We have used Dirichlet boundary condition for Electromagnetic field and MIT bag boundary condition for electron. To control ultraviolet divergences we have used dimensional regularization and a photon mass μ also is used to control infrared divergences. It is found that photonic loop does not have any contribution up to order α . Our result for fermionic loop, up to this order, have had a minor correction to the BRW result [34, 35] which they have considered only one diagram of three at this order. We have compared our results with the one they have obtained in Fig. 1.

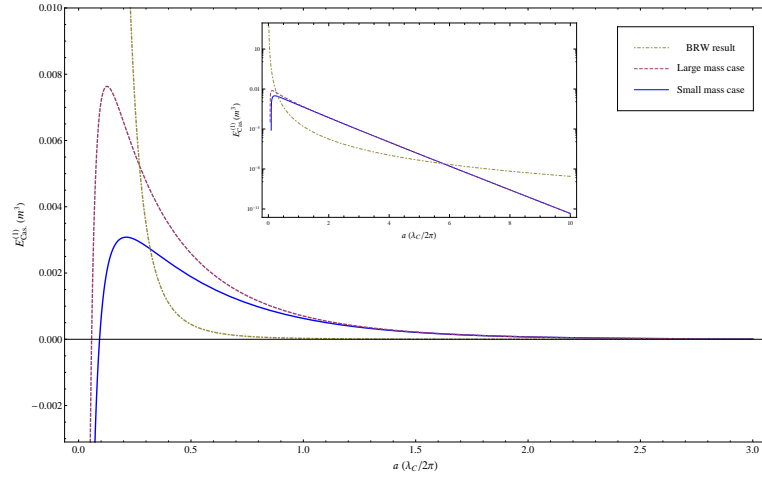


FIG. 1: The figure shows the $O(\alpha)$ radiative correction to the Casimir energy per unit area due to the fermionic and photonic loops vs the distance between plates (in terms of particle Compton wavelength) for two cases: small mass limit (solid line) and large mass limit (dashed line). Dot-dashed line is BRW result. Log-plot has been also depicted therein for comparison.

Appendix: Calculation Of The Branch-Cut Terms

In this Appendix we calculate two types of branch-cut terms which appear in Eqs. (29) and (42). If we note to Eqs.(29) and (42), regardless of some coefficients, they are completely the same. Hence we consider only branch-cut terms of Eq.(42).

$$\begin{aligned} \Delta E_F^{\text{Cas.}} = & \frac{e^2 L^2 m^2}{16\pi^2 a} \left\{ \sum_{n=0}^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + \left(n + \frac{1}{2}\right)^2} \left[-6 + 2\gamma + \ln \left(\frac{\frac{a^2 m^2}{\pi^2} + \left(n + \frac{1}{2}\right)^2}{\left(\frac{2a}{m^3}\right)^2} \right) \right] \right. \\ & - \int_0^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + p'^2} \left[-6 + 2\gamma + \ln \left(\frac{\frac{a^2 m^2}{\pi^2} + p'^2}{\left(\frac{2a}{m^3}\right)^2} \right) \right] \\ & + \sum_{n=0}^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + \left(n + \frac{1}{2}\right)^2} \left[10 - 6\gamma + 6 \ln 2 - 3 \ln \left(\frac{\frac{a^2 m^2}{\pi^2} + \left(n + \frac{1}{2}\right)^2}{\left(\frac{a\pi}{m^3}\right)^2} \right) \right] \\ & \left. - \int_0^{\infty} \sqrt{\frac{a^2 m^2}{\pi^2} + p'^2} \left[10 - 6\gamma + 6 \ln 2 - 3 \ln \left(\frac{\frac{a^2 m^2}{\pi^2} + p'^2}{\left(\frac{a\pi}{m^3}\right)^2} \right) \right] \right\} + O(\alpha^2). \end{aligned}$$

For the ease of calculation we use these change of variables: $b = -6 + 2\gamma$, $b' = 10 - 6\gamma + 6 \ln 2$, $A = \frac{e^2 L^2 m^2}{16\pi^2 a}$, $B = \left(\frac{2a}{m^3}\right)^2$, $B' = \left(\frac{a\pi}{m^3}\right)^2$ and $C = \frac{a^2 m^2}{\pi^2}$. We can apply (APSF)

$$\sum_{n=0}^{\infty} f\left(n + \frac{1}{2}\right) = \int_0^{\infty} dx f(x) - i \int_0^{\infty} \frac{dt}{e^{2\pi y} + 1} [f(it) - f(-it)].$$

Now, the integrand of the last term becomes

$$\begin{aligned}
f(it) - f(-it) &= \underbrace{Ab \left(\sqrt{C + (it)^2} - \sqrt{C + (-it)^2} \right)}_X \\
&+ \underbrace{A \left(\sqrt{C + (it)^2} \ln \left(\frac{C + (it)^2}{B} \right) - \left(\sqrt{C + (-it)^2} \ln \left(\frac{C + (-it)^2}{B} \right) \right)}_Y \\
&+ Ab' \left(\sqrt{C + (it)^2} - \sqrt{C + (-it)^2} \right) \\
&- 3A \left(\sqrt{C + (it)^2} \ln \left(\frac{C + (it)^2}{B'} \right) - \sqrt{C + (-it)^2} \ln \left(\frac{C + (-it)^2}{B'} \right) \right). \tag{53}
\end{aligned}$$

Choosing, $C = |C|e^{i\theta_C}$, $t = |t|e^{i\theta_t}$, we have for the first term

$$\begin{aligned}
X &= Ab \left(\sqrt{C + (it)^2} - \sqrt{C + (-it)^2} \right) \\
&= \left(\sqrt{|C|e^{i\theta_C} + (e^{i\pi}|t|^2e^{i2\theta_t})} - \sqrt{|C|e^{i\theta_C} + (e^{-i\pi}|t|^2e^{i2\theta_t})} \right) \\
&= \left(e^{\frac{i\pi}{2}}e^{i\theta_t} - e^{-\frac{i\pi}{2}}e^{i\theta_t} \right) \sqrt{|C|e^{i(\theta_C+\pi-2\theta_t)} + |t|^2} \\
&= 2iAb \sin \frac{\pi}{2} \sqrt{t^2 - C} \tag{54}
\end{aligned}$$

where one should note that $e^{i(\theta_C+\pi-2\theta_t)} = -1$. This term has been calculated for $t > \sqrt{C}$. For $t < \sqrt{C}$ this term is exactly zero. Similarly for second term, we have

$$\begin{aligned}
Y &= \sqrt{C + (it)^2} \ln \left(\frac{C + (it)^2}{B} \right) - \sqrt{C + (-it)^2} \ln \left(\frac{C + (-it)^2}{B} \right) \\
&= - \left(\sqrt{C + e^{i\pi}t^2} - \sqrt{C + e^{-i\pi}t^2} \right) \ln B \\
&+ \underbrace{\left(\sqrt{C + e^{i\pi}t^2} \ln(C + e^{i\pi}t^2) - \sqrt{C + e^{-i\pi}t^2} \ln(C + e^{-i\pi}t^2) \right)}_{Y'}. \tag{55}
\end{aligned}$$

The first term is exactly X (ignoring the overall factor), so we only need to calculate the second term:

$$\begin{aligned}
Y' &= \sqrt{C + e^{i\pi}t^2} \ln(C + e^{i\pi}t^2) - \sqrt{C + e^{-i\pi}t^2} \ln(C + e^{-i\pi}t^2) \\
&= \sqrt{C + e^{i\pi}t^2} \ln(e^{i\pi}(e^{-i\pi}C + t^2)) - \sqrt{C + e^{-i\pi}t^2} \ln(e^{-i\pi}(Ce^{i\pi} + t^2)) \\
&= i\pi(\sqrt{C + e^{i\pi}t^2} + \sqrt{C + e^{-i\pi}t^2}) + (\sqrt{C + e^{i\pi}t^2} - \sqrt{C + e^{-i\pi}t^2}) \ln(t^2 - C). \tag{56}
\end{aligned}$$

Now, the first term is again similar to X but with the plus sign between its terms, so we easily obtain

$$\sqrt{C + e^{i\pi}t^2} + \sqrt{C + e^{-i\pi}t^2} = 2\sqrt{t^2 - C} \cos \frac{\pi}{2}. \tag{57}$$

For the second term, using Eq.(54) and Eq.(57), we have for $t < \sqrt{C}$

$$\sqrt{C + e^{i\pi}t^2} \ln(C + e^{i\pi}t^2) - \sqrt{C + e^{-i\pi}t^2} \ln(C + e^{-i\pi}t^2) = 2i\sqrt{t^2 - C} \ln(t^2 - C). \tag{58}$$

For $t < \sqrt{C}$ this term is exactly zero. Similarly, third and forth terms in Eq. (53) is easily derived. Finally by

replacing values of C , B , B' , b , b' , Eq.(53) becomes

$$f(it) - f(-it) = \frac{e^2 L^2 m^2 \sqrt{-a^2 m^2 + \pi^2 t^2}}{8a\pi^2} \left[4 - 4\gamma - 2 \ln \left(\frac{-a^2 m^2 + \pi^2 t^2}{(2a\pi)^2} \right) \right]. \quad (59)$$

Hence our radiative correction becomes

$$\Delta E_F^{\text{Cas.}} = \int_{\frac{am}{\pi}}^{\infty} dt \frac{e^2 L^2 m^2 \sqrt{-a^2 m^2 + \pi^2 t^2}}{8a\pi^2 (e^{2\pi t} + 1)} \left(4 - 4\gamma - 2 \ln \left(\frac{-a^2 m^2 + \pi^2 t^2}{(2a\pi)^2} \right) \right) + O(\alpha^2).$$

-
- [1] H. B. G. Casimir and D. polder, The Influence of Retardation on the London-van der Waals Forces, *Phys. Rev.* **73** (1948) 360.
 - [2] M. J. Sparnaay, Measurements of attractive forces between flat plates, *Physica* **24**, 751 (1958).
 - [3] M. Bordag, U. Mohideen and V.M. Mostepanenko, New developments in the Casimir effect, *Phys. Rev.* **353** (2001), [arXiv:quant-ph/0106045].
 - [4] K. A. Milton, The Casimir Effect: Physical Manifestations of Zero-Point Energy, (World Scientific Publishing Co. 2001).
 - [5] S. K. Lamoreaux, Demonstration of the Casimir Force in the 0.6 to 6 m Range, *Phys. Rev. Lett.* **78**, 5 (1997).
 - [6] G. Bressi, G. Carugno, R. Onfrio, G. Ruoso, Measurement of the Casimir Force between Parallel Metallic Surfaces, *Phys. Rev. Lett.* **88**, 041804 (2002).
 - [7] A. Gusso, and A. G. M. Schmidt, Repulsive Casimir Forces Produced in Rectangular Cavities: Possible Measurements and Applications, *Brazilian Journal of physics* **36**, 1B 168 (2006).
 - [8] D. Garcia-Sanchez, K. Y. Fong, H. Bhaskaran, S. Lamoreaux, and H. X. Tang, Casimir Force and In Situ Surface Potential Measurements on Nanomembranes, *Phys. Rev. Lett.* **109**, 027202 (2012).
 - [9] W. J. Kim, A. O. Sushkov, D. A. R. Dalvit, and S. K. Lamoreaux, Surface contact potential patches and Casimir force measurements, *Phys. Rev. A* **81**, 022505 (2010).
 - [10] S. K. Lamoreaux, Reanalysis of Casimir force measurements in the 0.6-to-6-m range, *Phys. Rev. A* **82**, 024102 (2010).
 - [11] E. Elizalde, Matching the observational value of the cosmological constant, *Phys. Lett. B* **516**, 143 (2001).
 - [12] F. Bauer, M. Lindner, and G. Seidl, Casimir energy in deconstruction and the cosmological constant, *JHEP* **05**, 026 (2004).
 - [13] G. Mahajan, S. Sarkar, T. Padmanabhan, Casimir effect confronts cosmological constant, *Phys. Lett. B* **641** (2006).
 - [14] M. Fabinger, P. Horava, Casimir effect between world-branes in heterotic M-theory, *Nucl. Phys. B* **580**, 243 (2000).
 - [15] K. Poppenhaeger, S. Hossenfelder, S. Hofmann, M. Bleicher, The Casimir effect in the presence of compactified universal extra dimensions, *Phys. Lett. B* **582**, 1 (2004).
 - [16] L.P. Teo, Casimir effect in spacetime with extradimensions from Kaluza-Klein to Randall-Sundrum models, *Phys. Lett. B* **682** (2009).
 - [17] Roman Linares, Hugo A. Morales-Tecotl, Omar Pedraza, Casimir force in brane worlds: Coinciding results from Green's and zeta function approaches, *Phys. Rev. D* **81**, 126013 (2010).
 - [18] N. A. Kawakami, M. C. Nemes, and Walter F. Wreszinski, The Casimir effect for parallel plates revisited, *J. Math. Phys.* **48**, 102302 (2007).
 - [19] S. Hacyan, R. Jauregui, and C. Villarreal, Spectrum of Quantum Electromagnetic fluctuations in rectangular cavities, *Phys. Rev. A* **47**, 4204 (1993).
 - [20] H. Cheng, The Casimir energy for a rectangular cavity at finite temperature, *J. Phys. A: Math. Gen.* **35**, 2205 (2002).
 - [21] X. Li and X. Zhai, Rigorous proof of the attractive nature for the Casimir force of a p-odd hypercube, *J. Phys. A: Math. Gen.* **34**, 11053 (2001).
 - [22] G. J. Maclay, Analysis of zero-point Electromagnetic energy and Casimir forces in conducting rectangular cavities, *Phys. Rev. A* **61**, 052110 (2000).
 - [23] W. Lukosz, Electromagnetic zero-point energy and radiation pressure for a rectangular cavity, *Physica*, **56**, 109 (1971).
 - [24] J. R. Ruggiero, A. Villani and A. H. Zimerman, Some comments on the application of analytic regularisation to the Casimir forces, *J. Phys. A: Math. Gen.* **13**, 761 (1980).
 - [25] H. Ahmedov and I. H. Duru, New Casimir energy calculations for single cavities, *NATO Science Series* **213** (2006).
 - [26] M. Bordag, G. L. Klimchitskaya, U. Mohideen, V. M. Mostepanenko, *Advances in the Casimir effect*, Oxford University press (2009).
 - [27] M. A. Valuyan, R. Moazzemi and S. S. Gousheh, A direct approach to the electromagnetic Casimir energy in a rectangular waveguide, *Journal of Physics B*, **41**, 145502 (2008).
 - [28] P. A. M. Neto, The dynamical Casimir effect with cylindrical waveguides, *J. Opt. B: Quantum Semiclass.* **7**, s86 (2005).
 - [29] F. D. Mazzitelli, M. J. Sanchez, N. N. Scoccola, and J. von Stecher, Casimir interaction between two concentric cylinders: Exact versus semiclassical results, *Phys. Rev. A* **67**, 013807 (2003).
 - [30] D. A. R. Dalvit, F. C. Lombardo, F. D. Mazzitelli, and R. Onofrio, Exact Casimir interaction between concentric cylinders, *Phys. Rev. A* **74**, 020101 (2006).

- [31] M. Bordag, E. Elizalde, K. Kirsten, and S. Leseduarte, Casimir energies for massive scalar fields in a spherical geometry, *Phys. Rev. D* **56** , 4896 (1997).
- [32] C. M. Bender, and K. A. Milton, Scalar Casimir effect for a D-dimensional sphere, *Phys. Rev. D* **50** , 6547 (1994).
- [33] Mustafa zcan, Scalar Casimir effect between two concentric spheres, *Int. J. Mod. Phys. A*, **27** , 1250082 (2012).
- [34] M. Bordag, D. Robaschik and E. Wieczorek, *Ann. Phys.* **165** (1985) 192.
- [35] M. Bordag, and J. Lindig, Radiative correction to the Casimir force on a sphere, *Phys. Rev. D* **58** , 045003 (1998); [arXiv:hep-th/9801129] D. Robaschik, K. Scharnhorst and E. Wieczorek, Radiative corrections to the Casimir pressure under the influence of temperature and external fields, *Ann. Phys. (NY)* **174** , 401 (1987); M. Bordag and K. Scharnhorst, $O(\alpha)$ Radiative Correction to the Casimir Energy for Penetrable Mirrors, *Phys. Rev. Lett.* **81**, 3815 (1998); [arXiv:hep-th/9807121] S. S. Xue., Casimir effect of scalar field on $S(n-1)$ manifold, *Commun. Theor. phys. (Wuhan)* **11** , 243 (1989).
- [36] F. Ravndal and J.B. Thomassen, Radiative corrections to the Casimir energy and effective field theory, *Phys. Rev. D* **63** , 113007 (2001).
- [37] X. Kong and F. Ravndal, Radiative Corrections to the Casimir Energy, *Phys. Rev. Lett.* **79** , 545 (1997).
- [38] K. Melnikov, Radiative corrections to the Casimir force and effective field theories, *Phys. Rev. D* **64** , 045002 (2001).
- [39] R.M. Cavalcanti, C. Farina, and F.A. Barone, Radiative Corrections to Casimir Effect in the ϕ^4 Model, [arXiv:hep-th/0604200] (2006)
- [40] F. A. Barone, R. M. Cavalcanti, and C. Farina, Radiative corrections to the Casimir effect for the massive scalar field, [arXiv: hep-th/0301238v1](2003)
- [41] F. A. Barone, R. M. Cavalcanti, and C. Farina, Radiative corrections to the Casimir effect for the massive scalar field, *Nucl. Phys. Proc. Suppl.* **127** , 118 (2004). [arXiv:hep-th/0306011v2].
- [42] Reza Moazzemi, and Siamak S. Gousheh, A new renormalization approach to the Dirichlet Casimir effect for ϕ^4 theory in (1+1) dimensions, *Phys. Lett. B*, **658** (2008).
- [43] S.S. Gousheh, R. Moazzemi, M.A. Valuyan, Radiative correction to the Dirichlet Casimir energy for $\lambda\phi^4$ theory in two spatial dimensions, *Phys. Lett. B*, **681** (2009).
- [44] Reza Moazzemi, Maryam Namdar, Siamak S. Gousheh, The Dirichlet Casimir effect for ϕ^4 theory in (3+1) dimensions: A new renormalization approach, *JHEP* **0709**, 029 (2007).
- [45] L.H. Ford, *Proc. R. Soc. London A* **368** 305 (1979).
- [46] B.S. Kay, *Phys. Rev. D* **20** 3052 (1979).
- [47] D.J. Toms, *Phys. Rev. D* **21** 2805 (1980).
- [48] K. Langfeld, F. Schmuser, and H. Reinhardt, *Phys. Rev. D* **51** 765 (1995).
- [49] L.C. de Albuquerque, *Phys. Rev. D* **55** 7754 (1997).
- [50] Michael E. Peskin, Daniel V. Schroeder, *An introduction to Quantum Field Theory*, Addison-Wesley Publishing Company (1995).
- [51] P. Henrici, *Applied and computational complex analysis*, Vol. **1**, (Wiley, New York,1984), E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, (Cambridge University Press, 1958).
- [52] A.A. Saharian, arXiv:hep-th/0002239; arXiv:hep-th/ 0708.1187
- [53] P. N. Bogolioubov, *Ann. Inst. Henri Poincare* **8** , 163 (1967).
- [54] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, *Phys. Rev. D* **9** , 3471 (1974); A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, *ibid.* **10** , 2599 (1974).
- [55] R. Saghian, M.A. Valuyan, A. Seyedzahedi, and S.S. Gousheh, Casimir Energy For a Massive Dirac Field in One Spatial Dimension: A Direct Approach, *Int. J. Mod. Phys. A* **27** (2012) 1250038
- [56] A. Seyedzahedi, R. Saghian, and S. S. Gousheh, Casimir energy for a massless fermionic field confined inside a three-dimensional, *Phys. Rev. A* **82** , 032517 (2010).